A COMPARISON OF THE ONE-DIMENSIONAL RADIATIVE TRANSFER PROBLEM SOLUTIONS OBTAINED WITH THE MONTE CARLO METHOD AND THREE VARIATIONS OF THE DISCRETE ORDINATES METHOD

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Abstract. *Radiative transfer is the main phenomenon in the basis of several relevant problems of scientific and technological interest. Examples of application of the mathematical and computational modelling of such phenomenon can be found in astronomy, environmental sciences, engineering and medicine among many different areas. The integro-differential equation known as Boltzmann equation describes mathematically the interaction of the radiation with the participating medium, i.e. a medium which may absorb, scatter and emit radiation. Several methods have been developed for the solution of the Bolztmann equation. In the present work we present a comparison of the solutions obtained for the one-dimensional problem with four different methods: (i) Monte Carlo (MC) method; (ii) Discrete Ordinates method (S_N) combined with a finite difference approximation; (iii) Analytical Discrete Ordinates method (AS_N); and (iv) Laplace Transform Discrete Ordinates method (LTS_N). Our final objective is to solve the inverse radiative transfer problem and for that purpose we want to investigate methods that may provide accurate and fast solutions for the direct problem.*

Keywords: *: Radiative transfer, Boltzmann equation, Monte Carlo method, Discrete Ordinates Method, Laplace Transform Discrete Ordinates Method.*

1. INTRODUCTION

The formulation and solution of direct and inverse radiative transfer problems are directly related to several relevant applications in a large number of areas of scientific and technological interest such as tomography (Kim and Charette, 2007, Carita Montero et al., 2004), remote sensing and environmental sciences (Spurr et al., 2007, Verhoef and Bach, 2003, Hanan, 2001, Fause et al., 2001), and radiative properties estimation (Sousa et al., 2007, Silva Neto et al., 2007, Zhou et al., 2002), among many others.

Many approaches have been developed for the solution of such problems. Hansen and Travis (1974) and Lenoble (1977) provided excellent reviews on the methods for the solution of the direct radiative transfer problem, and McCormick (1992) did the same with respect to the inverse problem.

In recent years it has been observed a growing interest towards the stochastic Monte Carlo method for the solution of the direct problem (Maurente et al., 2007, Chen and Liou, 2006, Battaglia and Mantovani, 2005, Postylyakov, 2004, 2004a), as well as towards variations of the Discrete Ordinates Method (Çayan and Selçuk, 2007, Chalhoub, 2003, 2005) which was originally proposed by Wick (1943) and Chandrasekhar (1944, 1950). Moreover, some researchers have performed comparisons of different solution strategies in order to identify accurate and fast methods to be used both in the direct and inverse radiative transfer problems (Jensen et al., 2007, Bulgaretti and Doyle, 2004, Chalhoub et al., 2003).

In the present work we present a comparison of the solutions obtained for the direct radiative transfer problem in one-dimensional homogeneous and gray participating media with isotropic scattering using four different methods: (i) a Monte Carlo (MC) method; (ii) the Discrete Ordinates Method combined with a finite difference approximation, here denominated SMDO (Single Mesh Discrete Ordinates); (iii) the Analytical Discrete Ordinates method (SNA); and (iv) the Laplace Transform Discrete Ordinates Method (LTSN).

Our main objective is to investigate methods that can provide accurate and fast solutions for the direct problem in order to be used in the solution of the inverse radiative transfer problem.

2. THE TEST PROBLEM

In this work we consider a one-dimensional gray homogeneous, participating medium of optical thickness τ_0, with transparent boundary surfaces which are subjected to external radiation. It is assumed that the emission of radiation by the medium due to its temperature is negligible in comparison to the intensity of the external incoming radiation. Also the effects of possible differences on the refractive indices of the participating medium and surrounding environment are not taken into account. Our equation of transfer for such problem considering azymuthal symmetry and isotropic scattering within the medium is then given by (Özişik, 1973, Silva Neto and Moura Neto, 2005)

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu)+I(\tau,\mu)=\frac{\varpi}{2}\int_{-1}^{1}I(\tau,\mu')\mathrm{d}\mu', \qquad \tau\in(0,\tau_0), \quad \mu\in[-1,1], \quad \varpi\in[0,1] \tag{1}$$

subject to the boundary conditions

$$I(0,\mu)=f_1(\mu), \quad \mu>0, \qquad \text{and} \qquad I(\tau_0,\mu)=f_2(\mu), \quad \mu<0, \tag{2}$$

where $I(\tau,\mu)$ denotes the intensity (radiance) of the radiation field, τ the optical variable, μ the cosine of the polar angle, ϖ the albedo for single scattering, and f_1 and f_2 the intensity of the isotropic external sources of radiation incident at $\tau=0$ and $\tau=\tau_0$, respectively.

In order to solve the direct problem described by Eqs. 1 and 2, we use a Monte Carlo (MC) method and three variations of the Discrete Ordinates Method, originally proposed by Wick (1943) and Chandrasekhar (1944, 1950): SMDO - Single Mesh Discrete Ordinates; AS_N - Analytical Discrete Ordinates; and LTS_N - Laplace Transform Discrete Ordinates. These four methods, whose corresponding computational codes are referred to as MCPP, SMDO, PEESNA and LTSN, respectively, are described in the following sections.

3. THE MC METHOD

We present a summary of a Monte Carlo method that was based on the works of Cashwell and Everett (1959) and of Carter and Cashwell (1975). In this method, we adopted a physical approach that describes the transfer of radiation by following the history of many individual photons that are generated to represent a light source, until they are absorbed or escape the scattering medium. Quantities describing the photon initial position, the photon trajectories (such as direction of original emission, direction following scattering, and path length between interactions), and quantities describing interaction types (absorption or scattering) may be considered as random variables, each being characterized by some probability density function. In the following paragraphs, we show how to sample each one of the above mentioned quantities in order to track a photon as it penetrates into the considered medium.

The first required quantities are the position and direction of original emission (point sources), given in terms of the Cartesian coordinates x_0, y_0 and z_0 and the polar coordinates θ_0 and ϕ_0, with which we can calculate the first set of direction cosines

$$\Omega_0^x=\sin\theta_0\cos\varphi_0, \qquad \Omega_0^y=\sin\theta_0\sin\varphi_0, \qquad \text{and} \qquad \Omega_0^z=\cos\theta_0, \tag{3}$$

that are needed to determine the photon position at first collision.

The sampling of the photon paths length, performed by calculating the probability of a collision between the distances l and $l+\mathrm{d}l$ along its line of flight, is given by

$$p(l)\mathrm{d}l=e^{-\sigma_t l}\sigma_t\mathrm{d}l, \tag{4}$$

where $p(l)$ denotes the probability density function and σ_t the attenuation coefficient of the medium, which is interpreted as the probability per unit length of a collision. After setting

$$R_\tau=\int_0^l e^{-\sigma_t s}\sigma_t\mathrm{d}s=1-e^{-\sigma_t l}, \tag{5}$$

where R is the probability distribution function, we obtain the equation for the distance to collision, in optical length τ, as

$$l=-\frac{1}{\sigma_t}\ln(1-R_\tau)=-\frac{1}{\sigma_t}\ln R_\tau, \qquad \Longrightarrow \qquad \tau=l\sigma_t=-\ln R_\tau. \tag{6}$$

The new position can now be calculated by

$$x_2=x_1+\tau\Omega_1^x, \qquad y_2=y_1+\tau\Omega_1^y, \qquad \text{and} \qquad z_2=z_1+\tau\Omega_1^z, \tag{7}$$

where the subscripts 1 and 2 refer to the photon positions at subsequent collisions. With these new positions at hand, we are able to determine whether the particle is still within the system or escaped from it, in which case the sampling process is terminated.

In sampling the interaction types we define the probabilities

$$p_1 = \varpi = \frac{\sigma_s}{\sigma_t} \qquad \text{and} \qquad p_2 = \frac{\sigma_a}{\sigma_t}, \qquad \text{with } \sigma_t = \sigma_s + \sigma_a, \tag{8}$$

where ϖ denotes the single scattering albedo (or the probability of photon survival), σ_s the scattering coefficient and σ_a the absorption coefficient, and by drawing a random number R_ϖ we are able to determine the interaction type. So we let the interaction be an absorption event, considering the particle eliminated from the system and consequently the sampling process is terminated, when

$$R_\varpi \geq \varpi, \tag{9}$$

otherwise the interaction results in scattering.

The sampling of the scattering direction permits the estimation of the scattering angle through the use of the phase function $\tilde{\beta}(\theta, \phi)$. Here we consider that the phase function is only dependent on the scattering angle θ and that the azimuthal angle ϕ is uniformly distributed on the interval from 0 to 2π. Thus θ and ϕ become independent random variables that can be sampled separately. We also consider isotropic scattering, thus $\tilde{\beta}(\theta) = 1/4$. So by setting

$$R_\mu = 2\pi \int_{-1}^{\mu} \tilde{\beta}(\theta) \mathrm{d}\mu', \quad (\mu = \cos\theta), \qquad \text{and} \qquad R_\phi = \frac{1}{2\pi} \int_0^{\phi} \mathrm{d}\phi', \tag{10}$$

we obtain

$$\cos\theta = 2R_\mu - 1 \qquad \text{and} \qquad \phi = 2\pi R_\phi. \tag{11}$$

The new sets of direction cosines can now be calculated by the equations

$$\Omega_2^x = \sin\theta\cos\phi, \qquad \Omega_2^y = \sin\theta\sin\phi, \qquad \text{and} \qquad \Omega_2^z = \cos\theta, \tag{12}$$

The above sampling processes are repeated until the photon is absorbed or escapes the system under investigation. Radiometric quantities are computed by a suitable counting of photons through simulated detectors (counters) that are placed on the boundaries and layer interfaces. So by counting photons $I_c^{\pm}(\tau, \Delta\mu)$ traveling at a given location τ and within a given polar angle interval $\Delta\mu$, we are able to estimate the average radiance

$$I(\tau, \pm\mu) = \frac{S}{H} \frac{I_c^{\pm}(\tau, \Delta\mu)}{2\pi|\mu||\Delta\mu|}\ , \tag{13}$$

where S denotes the total source rates and H the number of photon histories, for $\mu \in [-1, 0)$ and $(0, 1]$, with μ averaged within the interval $\Delta\mu$.

4. THE SMDO METHOD

This method consists on a combination of the Discrete Ordinates Method with the finite difference method. First the angular domain is discretized as shown in Fig. 1, and the spatial domain is discretized as shown in Fig. 2.

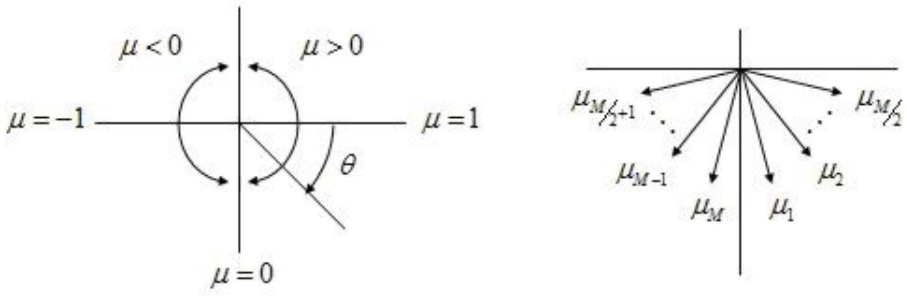

Figure 1. Angular domain discretization.

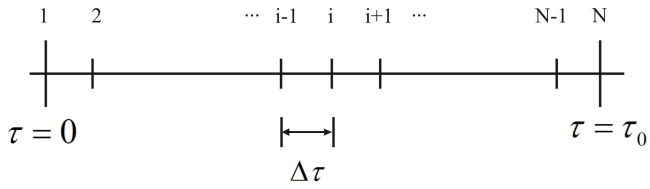

Figure 2. Spatial domain discretization.

The radiation intensities in the angular and spatial discretized domain are represented by

$$I_m^i = I(\tau_i, \mu_m), \quad \text{with } \tau_i = (i-1)\Delta\tau,\ i = 1, 2, \ldots, N \quad \text{and } m = 1, 2, \ldots, M, \qquad \Delta\tau = \frac{\tau_0}{N-1}. \tag{14}$$

The integral term on the right hand side of Eq. (1) is replaced by a Gauss-Legendre quadrature,

$$q^i = \frac{\varpi}{2}\int_{-1}^{1} I(\tau, \mu')\mathrm{d}\mu' \approx \frac{\varpi}{2}\sum_{n=1}^{N_q} a_n I_n^i \tag{15}$$

where a_n, $n = 1, 2, \ldots, N_q$, are the weights of the quadrature. The values of $\mu_m, m = 1, 2, \ldots, M$ ($M = N_q$), used in the angular domain discretization shown in Fig. 1, are the corresponding collocation points of the quadrature used.

Considering a forward and a backward finite difference discretizations of the first term on the left hand side of Eq. (1) given, respectively, by

$$\left.\frac{\partial I(\tau,\mu)}{\partial\tau}\right|_{(\tau_i,\mu_m)} = \frac{I_m^{i+1} - I_m^i}{\Delta\tau} \qquad \text{and} \qquad \left.\frac{\partial I(\tau,\mu)}{\partial\tau}\right|_{(\tau_i,\mu_m)} = \frac{I_m^i - I_m^{i-1}}{\Delta\tau}, \tag{16}$$

and from Eqs. (1) and (14)–(15) we obtain

$$I_m^{i+1} = \left(1 - \frac{\Delta\tau}{\mu_m}\right) I_m^i + \frac{\Delta\tau}{\mu_m} q^i \qquad \text{and} \qquad I_m^{i-1} = \left(1 + \frac{\Delta\tau}{\mu_m}\right) I_m^i - \frac{\Delta\tau}{\mu_m} q^i. \tag{17}$$

We performed forward (up to node N) and backward (back to node 1) sweeps, using the discrete boundary conditions expressed as

$$I_m^1 = f_1(\mu_m), \quad m = 1, 2, \ldots, \frac{M}{2} \qquad \text{and} \qquad I_m^N = f_2(\mu_m), \quad m = \frac{M}{2} + 1, \frac{M}{2} + 2, \ldots, M, \tag{18}$$

considering the following stopping criterion

$$\left|(I_m^i)^{k+1} - (I_m^i)^k\right| < \varepsilon, \qquad \text{with} \quad i = 1, 2, \ldots, N \quad \text{and} \quad m = 1, 2, \ldots, M, \tag{19}$$

where ε is a prescribed tolerance and k is the iteration index.

5. THE AS$_N$ METHOD

In this section, we present a summary of an improved version of the analytical discrete-ordinates method that has been the subject of some recent works (Barichello and Siewert, 1999; Barichello et al., 2000; Chalhoub and Garcia, 2000; Siewert, 2000). In particular, the method incorporates some recently developed techniques for finding particular solutions (Barichello et al., 2000; Siewert, 2000) and dummy-node inclusion (Chalhoub and Garcia, 2000) as its angular interpolation technique. Note that we only present here a simplified version for treating the type of problems described in section 2.

To define our discrete-ordinates version of the problem posed by Eqs. (1) and (2), we begin by introducing a quadrature of order N_q with nodes $\{\mu_j\}$ and weights $\{a_j\}$ to approximate the integral in Eq. (1). The selected quadrature scheme is the double quadrature of order $N_q = 2n$ obtained by applying a standard Gauss-Legendre scheme of order n to each of the half-intervals $[0, 1]$ and $[-1, 0]$. Then we set $\mu = \mu_j$, $j = 1, 2, \ldots, N_q$, in the resulting equations to find the discrete-ordinates equations

$$\mu_j \frac{\mathrm{d}}{\mathrm{d}\tau} I(\tau, \mu_j) + I(\tau, \mu_j) = \frac{\varpi}{2}\sum_{i=1}^{N_q} a_i I(\tau, \mu_i), \qquad j = 1, 2, \ldots, N_q, \tag{20}$$

and the boundary conditions

$$I(0,\mu_j) = f_1(\mu_j), \quad j = 1,2,\ldots,n, \qquad \text{and} \qquad I(\tau_0,\mu_j) = f_2(\mu_j), \quad j = n+1, n+2,\ldots,N_q. \tag{21}$$

Note that the nodes of the quadrature scheme are ordered in such a way that the first n nodes are *positive* and the remaining n are *negative*.

Making use of the elementary solutions of the discrete-ordinates equations and their orthogonality property developed in Barichello et al. 2000), we can write the general discrete-ordinates solution of order N_q to the problem formulated by Eqs. (20) and (21) as

$$I(\tau,\mu_j) = \sum_{k=1}^{n} A_k \Phi(\nu_k,\mu_j) e^{-\tau/\nu_k} + \sum_{k=n+1}^{N_q} B_k \Phi(-\nu_k,\mu_j) e^{-(\tau_0-\tau)/\nu_k} \tag{22}$$

for $j = 1, 2 \ldots, N_q$. The elementary solutions $\Phi(\nu_k,\mu_j)$ and $\Phi(-\nu_k,\mu_j)$ in Eq. (22) are, respectively, the jth components of the eigenvectors $\mathbf{\Phi}(\nu_k)$ and $\mathbf{\Phi}(-\nu_k)$, associated, respectively, with the eigenvalues $1/\nu_k$ and $-1/\nu_k$. Finally the coefficients $\{A_k\}$ and $\{B_k\}$ are the solutions to the linear system of N_q algebraic equations obtained by imposing that the general solution expressed by Eq. (22) satisfies the boundary conditions expressed by Eqs. (21)

$$\sum_{k=1}^{n} A_k \Phi(\nu_k,\mu_j) + \sum_{k=n+1}^{N_q} B_k \Phi(-\nu_k,\mu_j) e^{-\tau_0/\nu_k} = I(0,\mu_j), \tag{23a}$$

for $j = 1, 2, \ldots, n$, and

$$\sum_{k=1}^{n} A_k \Phi(\nu_k,\mu_j) e^{-\tau_0/\nu_k} + \sum_{k=n+1}^{N_q} B_k \Phi(-\nu_k,\mu_j) = I(\tau_0,\mu_j), \tag{23b}$$

for $j = n+1, n+2, \ldots, N_q$. We conclude this summary by pointing out that once the linear system formulated by Eqs. (23) is solved for $\{A_k\}$ and $\{B_k\}$, we can evaluate Eq. (22) for any $\tau \in [0, \tau_0]$.

6. THE LTS$_N$ METHOD

The LTS$_N$ scheme appeared in the early nineties in the neutron transport context [1], and was then extended to radiative transfer problems [30]. Its convergence was established using the C_0-semi group theory [30]. This method applies the Laplace transform on the radiative transfer discrete ordinates equation, Eq. (20). This yields a system of algebraic equations on s:

$$s\overline{I}^{\,m}(s) + \frac{1}{\mu_j}\overline{I}^{\,m}(s) = \frac{\varpi}{2\mu_j}\sum_{i=1}^{N} a_i \overline{I}^{\,m}(s) + I_j^m(0) \tag{24}$$

here: $\overline{I}^{m}(s) \equiv \int_0^\infty I^m(\tau) e^{-s\tau}\, d\tau$. The matrix form of equation (24) becomes

$$\overline{M}_N^{\,m}(s)\,\overline{I}^{\,m}(s) = I^m(0)\,, \qquad \text{being}: \ \overline{M}_N^{\,m}(s) = s\mathbf{I} + A^m \tag{25}$$

where the N-order matrix $\overline{M}_N^{\,m}(s)$ is called the LTS$_N$ matrix, and $\mathbf{I}$ is the N-order identity matrix. The entries of the A^m matrix are given by

$$A^m(i,j) = \begin{cases} \dfrac{1}{\mu_j} - \dfrac{\varpi\, a_j}{2\mu_j}\,, & \text{if } i = j, \\[2ex] -\dfrac{\varpi\, a_i}{2\mu_j}\,, & \text{if } i \neq j. \end{cases} \tag{26}$$

In order to solve the matrix equation (25), it must be multiplied by the inverse matrix of $\overline{M}_N^{\,m}(s)$, as follows

$$\overline{I}^{\,m}(s) = \left[\overline{M}^{\,m}(s)\right]^{-1} I^m(0) \quad \Leftrightarrow \quad \overline{I}^{\,m}(s) = \overline{B}^{\,m}(s)\, I^m(0)\,, \tag{27}$$

applying the Laplace inverse transform yields

$$I^{\,m}(\tau) = B^{\,m}(\tau) I^m(0)\,, \qquad \text{with}: \ B^m(\tau) \equiv \mathcal{L}^{-1}\left[\overline{B}^m(s)\right]\,. \tag{28}$$

Matrix inversion is usually expensive. The diagonalization method (see [31]) takes advantage of the fact that the LTS_N matrix, Equation (28), is non-degenerate, *i.e.* all eigenvalues are distinct, and therefore A^m can be diagonalized

$$A^m = X^m \, D^m \, (X^m)^{-1} \tag{29}$$

where D^m is a diagonal matrix containing the eigenvalues of A^m, and X^m is the corresponding eigenvectors matrix. Therefore, the matrix B can be expressed as

$$B^m(\tau) = \mathcal{L}^{-1}\left[(s\mathbf{I} + A^m)^{-1}\right] = X^m \mathcal{L}^{-1}\left[(s\mathbf{I} + D^m)^{-1}\right](X^m)^{-1} = X^m \, e^{D^m \tau}(X^m)^{-1} \, , \tag{30}$$

substituting Eq. (30) into Eq. (28), yields

$$I^{\,m}(\tau) = X^m e^{D^m \tau}(X^m)^{-1} I^m(0) \quad = X^m \left[\begin{array}{cc} e^{d^+\tau} & 0 \\ 0 & e^{d^-\tau} \end{array} \right]^m (X^m)^{-1} I^m(0) \tag{31}$$

where d^+ e d^- are positive and negative eigenvalues, respectively.

The method, as described in Eqs. (28) does not work, due to the numerical overflow for large slab thicknesses and/or large values of N. This feature can be avoided by changing the variables [18]. Equation (31) can be writen as follows

$$I^{\,m}(\tau) = X^m \left[\begin{array}{cc} e^{d^+(\tau-\tau_0)} & 0 \\ 0 & e^{d^-\tau} \end{array} \right]^m \left[\begin{array}{cc} e^{d^+\tau_0} & 0 \\ 0 & 1 \end{array} \right]^m (X^m)^{-1} I^m(0) = \mathcal{B}^{\,m}(\tau)\xi^m(0) \tag{32}$$

where:

$$\mathcal{B}^{\,m}(\tau) = X^m \left[\begin{array}{cc} e^{d^+(\tau-\tau_0)} & 0 \\ 0 & e^{d^-\tau} \end{array} \right]^m \; ; \; \text{and} : \; \xi^m(0) = \left[\begin{array}{cc} e^{d^+\tau_0} & 0 \\ 0 & 1 \end{array} \right]^m (X^m)^{-1} I^m(0). \tag{33}$$

Equation (32) can also be represented by block matrices:

$$\left[\begin{array}{c} I^1(\tau) \\ I^2(\tau) \end{array} \right]^m = \left[\begin{array}{cc} \mathcal{B}_{11}(\tau) & \mathcal{B}_{12}(\tau) \\ \mathcal{B}_{21}(\tau) & \mathcal{B}_{22}(\tau) \end{array} \right]^m \left[\begin{array}{c} \xi^1(0) \\ \xi^2(0) \end{array} \right]^m \tag{34}$$

with indexes 1 and 2 pointing to either right and left directions of radiances, respectively. This equation can be applied at the position $\tau = \tau_0$, allowing to compute the unknown values $[I^1(0)]^m$ for completing the $\mathrm{LTS_N}$ solution.

7. NUMERICAL RESULTS

Due to the good performance of the $\mathrm{AS_N}$ method in the comparisons of radiances generated by selected methods that was performed in a previous work (Chalhoub et al., 2003), we decided to use its generated results as reference values for the comparisons to be performed in this work.

Four problems, whose parameters are shown in Tab. 1, were chosen to perform the required comparisons. Besides these parameters we considered a quadrature order $N_q = 20$ for the SMDO, PEESNA and LTSN codes. In Table 2, we show the reference values which are the radiances $I(0, \mu)$ and $I(\tau_0, \mu)$ at selected values of μ ($M = 20$), generated by the PEESNA code for the chosen problems.

Table 1. Parameters used to define the problems.

Parameter	Meaning	Problem			
		1	2	3	4
ϖ	single scattering albedo	0.1	0.1	0.9	0.9
τ_0	optical thickness of the layer	0.5	5.0	0.5	5.0
f_1	intensity of external source at $\tau = 0$	1.0	1.0	1.0	1.0
f_2	intensity of external source at $\tau = \tau_0$	0.0	0.0	0.0	0.0

We note that a critical parameter in MCPP and SMDO codes had to be adjusted before performing the comparisons. For MCPP the critical parameter is the number of photon histories H and for SMDO it is the number of points in the spatial grid N. The greater the value of these parameters the more precise are the resulting radiances when compared to the reference values. Table (3) shows for MCPP the number of photon histories H, where, for example, 1K= 10^3 and 1M= 10^6 histories, and for SMDO the grid points N, used to reach the established precision. This table only shows the

Table 2. Radiation intensity $I(0,\mu)$ and $I(\tau_0,\mu)$ generated by PEESNA for the chosen problems.

μ	Problem 1		Problem 2		Problem 3		Problem 4	
	0	τ_0	0	τ_0	0	τ_0	0	τ_0
−.9983	1.2023E−2	0.0	1.6460E−2	0.0	1.6997E−1	0.0	4.1359E−1	0.0
−.9830	1.2317E−2	0.0	1.6764E−2	0.0	1.7406E−1	0.0	4.1812E−1	0.0
−.9426	1.2873E−2	0.0	1.7333E−2	0.0	1.8180E−1	0.0	4.2639E−1	0.0
−.8765	1.3750E−2	0.0	1.8208E−2	0.0	1.9397E−1	0.0	4.3865E−1	0.0
−.7864	1.5047E−2	0.0	1.9460E−2	0.0	2.1190E−1	0.0	4.5529E−1	0.0
−.6750	1.6937E−2	0.0	2.1206E−2	0.0	2.3786E−1	0.0	4.7683E−1	0.0
−.5451	1.9725E−2	0.0	2.3639E−2	0.0	2.7578E−1	0.0	5.0408E−1	0.0
−.4003	2.3970E−2	0.0	2.7110E−2	0.0	3.3240E−1	0.0	5.3823E−1	0.0
−.2446	3.0686E−2	0.0	3.2347E−2	0.0	4.1782E−1	0.0	5.8140E−1	0.0
−.0823	4.1229E−2	0.0	4.1435E−2	0.0	5.2819E−1	0.0	6.3885E−1	0.0
.0823	1.0	2.1375E−2	1.0	6.6726E−5	1.0	3.1047E−1	1.0	2.3734E−2
.2446	1.0	1.3295E−1	1.0	8.3119E−5	1.0	4.3493E−1	1.0	2.9994E−2
.4003	1.0	2.8155E−1	1.0	1.1019E−4	1.0	5.4492E−1	1.0	3.6249E−2
.5451	1.0	3.9250E−1	1.0	2.0825E−4	1.0	6.2029E−1	1.0	4.2794E−2
.6750	1.0	4.7044E−1	1.0	6.0521E−4	1.0	6.7145E−1	1.0	4.9672E−2
.7864	1.0	5.2516E−1	1.0	1.5370E−3	1.0	7.0675E−1	1.0	5.6675E−2
.8765	1.0	5.6350E−1	1.0	2.9783E−3	1.0	7.3123E−1	1.0	6.3347E−2
.9426	1.0	5.8977E−1	1.0	4.6397E−3	1.0	7.4788E−1	1.0	6.9123E−2
.9830	1.0	6.0657E−1	1.0	6.1245E−3	1.0	7.5849E−1	1.0	7.3484E−2
.9983	1.0	6.1547E−1	1.0	7.0790E−3	1.0	7.6410E−1	1.0	7.6038E−2

Table 3. Critical parameters and CPU times (s).

Problem	MCPP		SMDO
	H	CPU (s)	N
1	1M	1.5	50
2	100M	266.0	1000
3	10M	1.5	10
4	1M	53.0	200

CPU times for the MC method, as for the other methods the CPU times are less than 0.1 seconds. Note that the codes were executed on an IBM compatible personal computer equipped with a Pentium M 1.7GHz processor.

To illustrate how the critical prameters were chosen, Fig. 3 shows results obtained by running problem-3 with MCPP and SMDO codes using four different values of H and N, respectively. The E values, shown in this figure, represent a global percent deviation that were calculated by a modified version of the Euclidean metric equation

$$E = \frac{100}{M}\sqrt{\sum_{i=1}^{M}\left(\frac{p_i - q_i}{q_i}\right)^2}, \tag{35}$$

where p_i, $i = 1, 2, \ldots, M$, denote the radiances generated with a given critical value and q_i, $i = 1, 2, \ldots, M$, those generated with a higher critical value. We also note that for the performed comparisons we chose the critical values that generated results with $E \leq 1\%$.

In Figs. 4–5 we show the radiances generated by the four codes, as well as the E values that represent the global percent deviation in the radiances generated by each one of the codes from the reference values generated by the PEESNA code.

8. CONCLUSIONS

From the comparisons of the radiance generated by MCPP, SMDO, PEESNA, and LTSN codes, we conclude the following:

- As expected, the Monte Carlo method is the most expensive numerical procedure when compared with determinstc

techniques.

- The SMDO code requires some analyses to find out the ideal critical parameter, needing a preprocessing scheme.
- AS_N and LTS_N are semi-analytical methods and their solutions are exact for the space variable, as there are no intrinsic truncated errors.

The inverse radiative transfer problems can be formulated as an optimization problem, where implicit procedure can be used. Such strategy requires the solution of the forward problem many times up to convergence, therefore one important feature is to identify the most accurate and fast solutions for the direct problem. The results shown here allow us to say that any one of the used codes: SMDO, PEESNA, and LTSN, is a good choice for these requirements, considering an isotropic and homogenous medium, and without a source term. However, if someone is interested in testing some inverse procedures, one can use the Monte Carlo method output to represent the experimental measurements of minimizing the *inverse crime*.

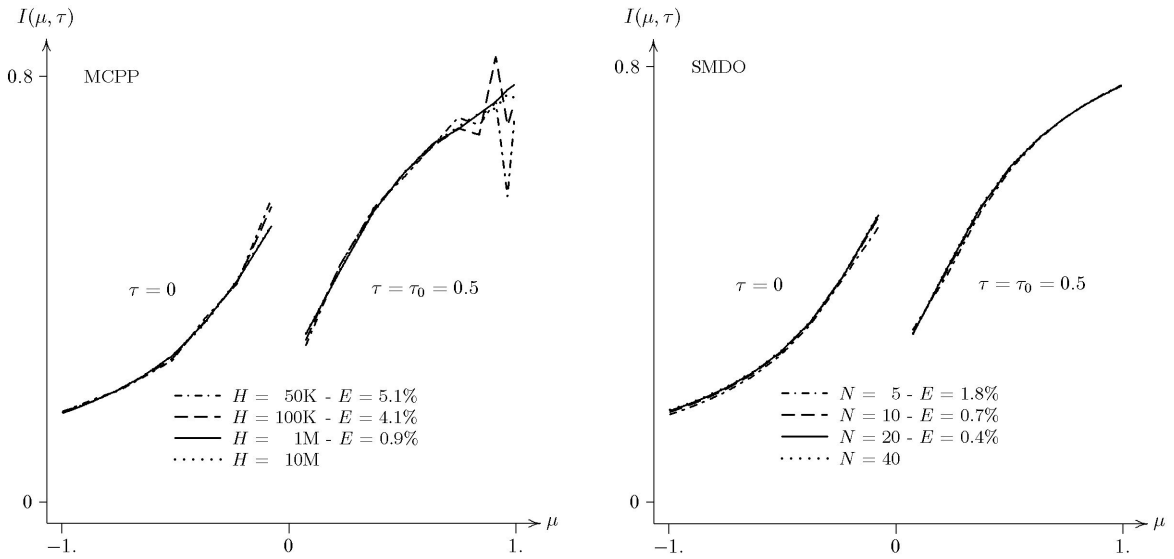

Figure 3. Radiance generated by MCPP and SMDO with $\varpi = 0.9$ and $\tau_0 = 0.5$.

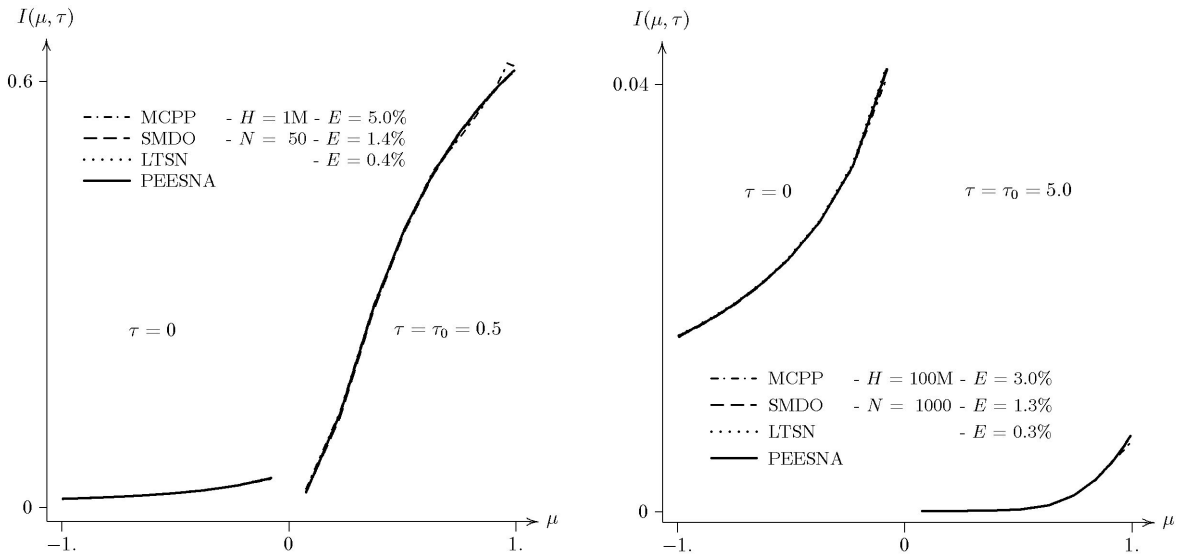

Figure 4. Radiance generated by the four codes with $\varpi = 0.1$ and $\tau_0 = 0.5$ and 5.0.

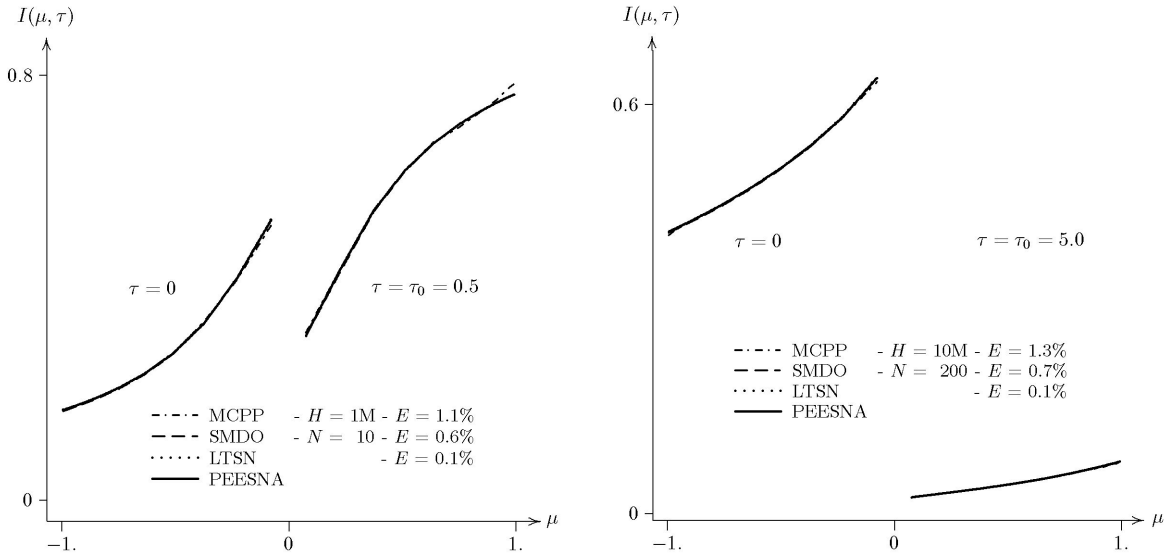

Figure 5. Radiance generated by the four codes with $\varpi = 0.9$ and $\tau_0 = 0.5$ and 5.0.

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